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Hindsight Technique in Machine Translation of Natural Languages

Ida Rhodes and Franz L. Alt

(December 21, 1961)

In the proposed system for automatic syntactic analysis of Russian sentences developed at the National Bureau of Standards, the computer splits each Russian word into stem and ending and combines the information obtained from these two elements into a morphological description of the word, frequently containing several alternatives. The decision among such alternatives is normally made on the basis of "predictions" arising from preceding words of the same clause. There are, however, cases in which no prediction is available to account for a word, e.g., when the object of a verb occurs before the verb itself. In such a case, instead of the usual prediction of the object, we need "hindsight." Also, it may happen that more than one of the morphological alternatives of a word agrees with predictions; or that a single morphological alternative agrees with several predictions; or that only one of them agrees, yet there is a suspicion that the agreement is spurious; or that no agreement at all is found. It turns out that the alternating use of prediction and hindsight techniques overcomes most of these troubles.

1. Introduction

The present paper may be considered as a progress report on a research project aimed at Russian-to-English translation by means of digital computers, which has been conducted at the National Bureau of Standards since 1959. The project differs from half a dozen others concerned with automatic translation from the Russian in its emphasis on conventional grammar. Some of the other researchers in this area have been concerned with the compilation of automatic dictionaries, with statistical studies of the frequency of occurrence of given words or constructions, with semantic problems, etc.; still others, though concerned with grammatical problems, have stressed the development of new theories of linguistic structure. By contrast, we start from the traditional grammatical concepts, as taught in school, and modify them only occasionally where this seems expedient for computer coding. Thus our first aim is the automatic syntactic analysis ("parsing") of Russian sentences.

The main features of this approach have been described in an earlier paper [1].[1] The terminology and notation of that paper have been retained in the present one, with a few minor changes due to refinements adopted since the former was written. For the convenience of the reader we repeat here a few salient points of that paper, amplified by examples which lead up to the subjects of the present paper.

Reduced to simplest terms, our translation system rests on the following concepts. A Russian word, in general, is capable of several grammatical interpretations. For instance, the Russian word нации may be the genitive, dative or locative singular, or nominative or accusative plural, of the noun нация (nation). For another example, an adjective ending in —ым may be either in the dative plural, any gender, or in the instrumental singular, masculine or neuter gender. We call these alternatives "temporary choices" (TC). Furthermore, a word frequently calls for certain other words or forms of words. For instance, any form of the verb служить (to serve) is likely to be followed by a noun in the dative and/or by a noun in the instrumental case; similarly, a transitive verb predicts an object in the accusative case; many verbs, nouns, or adjectives predict instrumentals; at the start of each clause in a sentence we predict a subject and a predicate. Information of this kind is called "foresight predictions" (FP). The proposed translation scheme proceeds by choosing one among several alternative TC's for a word on the basis of previously recorded FP's; the chosen TC is labeled "selected choice" (SC). (In examining FP's, the most recent one is taken first; and the first agreement encountered is used as SC.) Thus, if the words служат нации occur in a sentence in this order, the latter word will be considered as dative singular, and will be translated accordingly (unless this decision is superseded by information obtained elsewhere in the sentence). In the normal course of events the computer scans a sentence from left to right, one word (or "occurrence") at a time, assigns an SC to each word and, based on it, forms additional FP's which are stored in the machine for use with subsequent words. The left-to-right scanning of a sentence may be iterated if necessary.

A number of complications can arise. For example, the object of a verb may occur before the verb itself; instead of the usual prediction of the object we need "hindsight" to explain its occurrence. In general, the appearance of a word before its prediction is quite frequent. Incorporating this feature into the machine code raises interesting problems. Another complication enters when more than one of the morphological alternatives of a word agrees with predictions; or when only one of them agrees, yet there is a suspicion that the agreement is spurious; or if no agreement at all is found. It turns out that the alternating use of prediction and hindsight techniques overcomes most of these troubles.

[1] Figures in brackets indicate the literature references at the end of this paper.

2. Definitions

We see then that the hindsight techniques are used in certain cases where the usual process of choosing a selected choice (SC) from among the temporary choices (TC) on the basis of foresight predictions (FP) runs into some kind of trouble, and where there is reason to expect that subsequent words of the same sentence will help to explain the situation. In such cases, rather than search for the explanation at once, the machine stores certain information in one of several memory areas set aside for this purpose. This information is called a hindsight entry (HE). Information obtained from subsequent words which sheds partial light on a particular HE is called a hindsight resolution (HR) and is stored alongside the HE. Whenever an occurrence completely resolves a difficulty which has caused an HE, this is called an explanation or decision. When this occurs, the HE and any previous HR's pertaining to it are erased.

We distinguish four types of hindsight, designated $H_0, \ldots, H_3$. Each has its own reserved area in the machine's memory, its own entries, labeled H_0E, etc., and its own resolutions, labeled H_0R, etc.

Briefly and with some oversimplification, H_0 is used when *no* agreement is found between any TC of the current word and any FP on record; H_2 is used when *two* or more agreements are found; H_1 is used when there is an agreement which, however, is considered doubtful and likely to be superseded by information obtained later in the sentence. Finally, H_3 includes all those TC's which have served neither as SC nor as HE. However, in some cases we discard these left-over TC's without even recording them in H_3; namely, when the SC fits so well that there is no doubt about its being the correct choice. Examples of this are given in section 6.

The foregoing definitions of types H_0 to H_3 have to be modified for so-called "unpredictable" occurrences. This will be discussed in the following sections.

Resolutions may be obtained at several stages of the examination of an occurrence during syntactic analysis:

1. After the new foresight predictions are made, yielding H_iR_{FP} $(i=0, 1)$;
2. after the TC's of an occurrence are compared with the FP's and the selected choice is made, yielding H_iR_{SC} $(i=0, 1, 2)$;
3. at the completion of the processing of one occurrence, when the unselected TC's are examined for possible clues, yielding H_1R_{TC}.

Thus, the four types of hindsight differ both in respect to the situations from which they arise, and in respect to the circumstances which can contribute to their resolution. As we have just stated, a hindsight entry of type 0, H_0E, admits resolutions from FP and SC; an H_1E admits resolutions from FP, SC and TC, and H_2E only from SC. No resolutions are recorded for H_3 entries.[2] The reasons for these differences will become clearer in the next few sections.

[2] These entries are used only in subsequent iterations of the process.

It may happen, of course, that an agreement between foresight prediction and temporary choice is established uniquely and — seemingly — without doubt, and therefore no hindsight entry is made, yet the agreement is invalidated by some information occurring later in the sentence. In such a case, a better translation is obtained in a later iteration of the left-to-right scanning of the sentence. Thus, hindsight serves the purpose of reducing the number of iterations. It is an economical expedient, rather than a logical necessity.

In the following we shall discuss some features of the hindsight scheme in more detail. This discussion should be understood as a collection of illustrations, rather than as a complete treatment. It is based on small samples of text, and will undoubtedly undergo considerable revision in the course of time.

3. Hindsights of Type 0

If, in the analysis of a given word of a sentence, we find that none of its TC's satisfy any of the FP's on record, then in general we choose the first TC as SC, and make a record of the situation in H_0E. The listing of TC's in the machine is roughly in order of the frequency with which they occur, so that in the absence of other information the first one is probably the best choice.

Certain TC's are considered unpredictable; for instance, prepositions, conjunctions, adverbs, and certain idioms taking the place of adverbs (incidental expressions, parenthetic remarks.) Similarly, accusatives of nouns designating time, certain instrumentals, and the gerunds of certain verbs are considered unpredictable. If all TC's of one occurrence are unpredictable, we choose the first one but make no entry in H_0. (There would be no point to such an entry, since no resolution is either possible or needed.)

If some TC's of an occurrence are predictable while others are not, the predictable ones are examined first. If no FP matches any of them, the first unpredictable TC is chosen as SC and no entry in H_0 is made. However, the commonest of these cases, such as the ambiguity between adverb and short-form neuter adjective (adjectival stem with ending —o) are entered in H_1.

Inasmuch as an H_0E records the fact that an SC has been chosen without the required FP, it is the purpose of the resolutions to find an FP which explains either the SC or an alternative TC. There are certain kinds of TC which cannot possibly be governed by a subsequent occurrence, and such TC's may be deleted at the time H_0E is made. For instance, a locative case can only be called for by a preceding preposition (or as "master" by a preceding adjective which in turn is governed by a preposition). In the example given in the introduction, the word нашим has the TC's: *gen., dat., loc. sing.;* and *nom., acc. plur.;* if this word occurs at a time when there is no prediction of a locative on record, the TC *loc.* may be deleted. See, however, section 7(c).

Subsequent to the recording of an H_0E, every new FP is compared with the TC's of the H_0E and, if satisfied by one of them, is recorded as a "partial resolution for H_0 at time FP," H_0R_{FP}. This is in addition to its recording in the foresight pool as usual. If a subsequent word has a TC which satisfies the same FP, this fact is marked alongside the H_0R_{FP}. This marking is designated H_0R_{SC}.

4. Hindsights of Type 1

As we indicated in section 2, a hindsight entry of Type 1 (H_1E) is made when, in comparing the TC's of one occurrence with the FP's recorded in the foresight pool, an agreement is found, and when this agreement is of a kind considered doubtful. We have a standard list of such doubtful cases of agreement. Examples are:

A word which could be either nominative or accusative chosen as subject.

A verb in the infinitive chosen as subject.

A short-form adjective, neuter singular, which could also be an adverb.

The word и chosen as a conjunction; it could also be an adverb or part of the pair и . . . и.

One of the words его, ее, их, which may be rendered in English either as pronouns or as possessive adjectives.

Since H_1E is based on a match between FP and TC, resolutions must provide alternative explanations both for the TC and for the FP; these can be furnished by subsequent FP's and TC's, respectively. Thus a subsequent FP which is satisfied by the TC of H_1E or by one of its alternate TC's, is recorded as a resolution H_1R_{FP}, while a subsequent TC which satisfies the FP of H_1E is recorded as H_1R_{TC}. In addition, if the FP of an H_1R_{FP} is also satisfied by a later TC, this fact is marked in an H_1R_{SC}. In the case of the FP "Subject" or "Predicate," the fact that subject and predicate must agree in person, gender and number may sometimes eliminate some of the competing matches.

Obviously, a complete resolution is unattainable unless at least one resolution by FP *and* one by TC are recorded.

5. Hindsights of Type 2

Sometimes the process of comparing the TC's of an occurrence with the FP's in the foresight pool results in more than one agreement. We may find one FP satisfied by several TC's, or one TC satisfying several FP's, or different TC's in agreement with different FP's. The last case is the most obvious one. Thus in the example given in the introduction, the form нашим has five TC's (*gen., dat., loc. sing.*, and *nom., acc. plur.*). If this word occurs at a time when the foresight pool contains predictions of accusative and genitive complements, these two FP's are satisfied by two different TC's of the present occurrence. An instance of one FP satisfied by two TC's occurs when, say, an adjective ending in —ом, designating the locative singular of either masculine or neuter gender, is encountered at a time when the foresight pool contains a prediction of a complement in the locative case. Two FP's satisfied by the same TC might be predictions of a genitive complement and of an adjectival modifier in the genitive case; this case is discussed under (a) below.

Whenever we have several agreements between FP's and TC's, the first such agreement found is chosen as SC; the others are, in general, entered in H_2E. (Exceptions will be discussed in sec. 6). Numerous combinations of SC and alternative satisfiable FP's are possible. As was done for H_1, these possible combinations will be numbered in accordance with a standard list. At the present time only a few examples of items on this list can be given; other cases will be added to the list as they are encountered in text.

(a) SC=complement, genitive; alternative FP=modifier, genitive. This situation arises when a genitive noun is followed by an adjective which agrees with the noun: нахождение тела максимального объема. The noun тела, in accordance with the rules of grammar, predicts a possible complement in the genitive case, and a possible adjectival modifier. The following adjective максимального satisfies both predictions. In such a case we choose "complement" as SC and record "modifier" in H_2E. The adjective, in turn, predicts a "master," i.e., a noun or other declinable word agreeing in case, number, gender, and animation. (Incidentally, the adjective максимального has a third TC, namely *acc. sing. masc. anim.*; if there is no prediction of an accusative on record, this TC would merely be entered in H_3.) For resolutions to be used with this case, the following empirical rule appears to work. If the "master" prediction is satisfied by a subsequent occurrence, this constitutes a complete decision. Since the adjective has seemingly found a master, we assume it cannot also serve as modifier of the preceding noun. Thus the originally chosen SC is confirmed, and the H_2E is erased. If, on the other hand, no master is found, the situation is reversed and the SC for the adjective is considered to be the postpositive modifier (attribute), unless it is marked as the kind of adjective which is frequently used as a noun. The other alternative choice is left standing in H_2E. Barring the remote possibility that the ambiguity will somehow be resolved by a subsequent occurrence, both versions will be printed out. Note that adjectives used as nouns sometimes require a different English translation. Thus, in вычисление объема последнего окончено ("the computation of the volume of the latter is completed") the adjective последнего is first considered as complement, with the alternative "modifier" stored in H_2. The SC "complement" predicts a master. When no master is found, we consider interchanging SC and H_2E; since, however, последний is often used as a noun and is so marked in the dictionary, we decide to retain the SC "complement," leaving "modifier" in H_2, and assigning the noun meaning to it. As such it is translated "latter," while as an adjective the likely translation is "last."

(b) An adjective which can be genitive singular for masculine or neuter gender, animate or inanimate; or else accusative singular for masculine animate (i.e., adjective ending in —oro or —ero). If such an adjective occurs at a time when there are extant predictions both for accusative and genitive complement, one of the cases (the first one found to give agreement) is chosen as SC, the other goes to H_2E. This entry may be resolved in more than one way. If the adjective is followed by a "master" which is inanimate, then it could not have been the accusative; this would be considered a "decision." If a subsequent occurrence of the same clause, other than the master, is either clearly an accusative or clearly a genitive, it can be used to satisfy one of the two competing predictions, leaving the other one to be assigned to the ambiguous adjective. Such a resolution is entered as H_2R_{SC}. For example: Мы читали на курсах знаменитого учителя старую книгу, изданную в 1753 ("we read in the class of the famous teacher an old book, published in 1753"). By the time we arrive at знаменитого ("famous"), there are predictions of an accusative object, governed by the verb "read," and of a genitive complement, governed by the noun "class." The adjective знаменитого can be either *gen.* or *acc.* The machine assigns the former as SC; the latter is stored in H_2E. Next, this adjective predicts a master. If the following noun were an inanimate genitive, it would indicate that the adjective was also genitive. Since, however, учителя is animate, it can be either genitive or accusative, and thus throws no light on the ambiguity. If the sentence ended after учителя, the SC "*gen.*" and the H_2E "*acc.*" would both stand, and two translations would be printed; "we read in the class of the famous teacher" and "we read in the class the famous teacher." (If instead of читали we had a verb which *must* be accompanied by an accusative object, such as увидеть, the first version would be omitted.) Since, however, the sentence does not end here and the following words старую книгу are clearly accusative, this fact is entered as H_2R_{SC}, to serve as partial explanation. If, as is the case here, the remainder of the sentence throws no further light on the ambiguity, only the genitive version of the translation will be printed.

6. Omission of Hindsight Entries

In a number of cases we deviate from the general rules for recording hindsight entries, laid down in the preceding sections. This is done primarily if the selected choice appears secure beyond any doubt, but also when an alternative, if it were entered in hindsight, could not possibly lead to a resolution.

(a) The SC is undoubted:

(a–1) The SC is a prepositional complement. In this case no entry is made either in H_2 or in H_3. (Of course, items for entry in H_0 or H_1 do not even arise in this case.) In Russian, unlike English, a preposition must be followed by its complement, and nothing can intervene between the two except adverbs or certain incidental expressions. There are, however, cases where we have two FP's of prepositional complements, or where one preposition governs several complements; in such cases Rule (a–1) must be modified. This will be discussed in the next section.

(a–2) The SC is the "master" of a preceding adjective.

(b) No resolution possible or necessary:

(b–1) A TC other than the SC is in the locative case. There is no need to enter this in any hindsight. The reason is that the locative can occur only as a prepositional complement (or as a master of adjectival modifier of another preceding locative). However, the case of multiple prepositional complements, to be discussed below, presents some complications.

(b–2) An unpredictable TC. Cases of unpredictable TC's are enumerated in section 3 above. There is no point in making an H_0 entry, since the only purpose of such an entry is to allow for a subsequent prediction. The case cannot arise for H_2, since this would presuppose an existing prediction. Some frequent ambiguities involving unpredictable TC's are among the standard types of H_1, such as that of the short-adjective/adverb ending in —o. If another TC has been chosen as SC, and if the case is not in H_1, then the unpredictable TC is entered in H_3.

7. Multiple Prepositional Complements

Special treatment is required in certain cases where a preposition can govern different cases, and thus generates several predictions of prepositional complement, or where a preposition governs a list of nouns or other declinables, all in the same case but possibly widely separated.

(a) The case where several predictions of prepositional complements are satisfied by the same occurrence happens most frequently with adjectives. For instance, the adjectival feminine ending —ой can be genitive, dative, instrumental, or locative, singular. Should a preposition such as с require either of a pair of these cases, we should have no way of determining which of the TC's is actually to be taken as the SC. In such a case, we shall choose *both* for the SC, and wait for the master to resolve the situation.

(b) The positional prepositions (requiring either the accusative or locative) may cause even greater trouble because there are some nouns whose endings do not distinguish between the locative singular and the accusative plural. An attempt is made to resolve this difficulty by storing a flag to indicate whether a previous occurrence governs a positional preposition and which of the cases would then be required. For example: the word основанный requires a на with a *locative*, whereas обращать внимание requires на with the accusative. Should either of these locutions be followed by a word, such as теории, which is either locative singular or accusative plural (among other cases), the above signals will indicate which is to be chosen.

(c) Occasionally, a locative may occur in a list of locatives governed by the same preposition (unrepeated). We believe that the profiling scheme [2] would in some cases be able to detect this situation, but we have not yet found a formula which would be successful in all situations. For example: Мы говорили о теории Фадеевой, очень интересной части высшей алгебры. (We were speaking about the theory of Fadeeva, a very interesting part of higher algebra.) The adjective интересной (interesting) is here locative and, together with the noun теории (theory) is governed by the preposition о (about), but it has many other TC's, some of which may agree with extant predictions. During the process of profiling, this adjective will be marked as possible head of an adjectival phrase beginning at the comma, and will be given a "backward flag," a machine indication of the possibility that it may be a modifier of an earlier noun. But this does not enable us to decide whether the adjective интересной agrees with the locative noun теории or with the genitive noun Фадеевой.

We remark parenthetically that examples like the last one can give rise to still further difficulties. The word Фадеевой may be interpreted as an adjective, modifying the preceding noun теории, rather than as a noun serving as genitive complement to теории (translated "the Fadeevian theory" or "the Fadeev theory"). If so, the ambiguity in интересной disappears, since there is now only one noun (теории) which it can modify. Furthermore the noun части (part) may be construed either as "master" of the preceding adjective интересной, agreeing with it in case, number and gender (an interesting part); or as a dative complement to the same adjective ("interesting to a part"). Also, the adjective высшей ("higher") may be construed as modifying either the noun preceding it or the one following it, and this alternative may be combined with any of the uses already enumerated. In addition, the noun части, instead of being construed as either the master or the dative complement of интересной, may be understood as a genitive complement to that preceding noun which also governs интересной. Another possibility is that the noun алгебры is the genitive of comparison after the comparative adjective высшей (giving the translation "higher than algebra"). Finally, there are several other minor types of unlikely translations. One such type involves a dependent genitive construction *preceding* its governor, as in "about the Fadeeva of the theory," "a higher algebra of a very interesting part," "a very much higher algebra of an interesting part," etc. Another such type is concerned with the fact that almost all Russian adjectives can act as nouns, requiring the insertion of the English word "one," and can then take a dependent genitive construction, as in "a higher one of algebra," "the theory of the Fadeevan one," etc. Furthermore, each noun, or adjective used as noun, can be considered to be an appositive to a preceding noun in the same case, yield ng constructions translated as "the theory Fadeeva," "the very interesting one, a part," etc. A small number of the several hundred possible syntactic interpretations of the sentence will be found below. A resolution of some of these ambiguities will be possible only after the formidable problems of semantics have been attacked.

Our scheme will print out the following as the most likely translation: We were speaking about the theory of Fadeeva, a very interesting part of higher algebra.

Unfortunately, our syntactic analysis will be incorrect, even though this is not shown in the translation, because English is not sensitive to case distinctions in nouns and adjectives. Here, our scheme will connect the word интересной appositively with the closest preceding noun which agrees in case with it, namely with Фадеевой, rather than with the correct noun теории.

Some further possible syntactic interpretations of the sentence are given below. Explanatory words and punctuation marks have been added to show the syntactic structure.

We were speaking about:

The theory of Fadeeva, who is a very interesting part of higher algebra.

The Fadeevan theory, which is a very interesting part of higher algebra.

The Fadeevan theory, who is a very interesting part of higher algebra.

The theory of the Fadeevan one, which is very interesting to a part of higher algebra.

The theory of the Fadeevan one, who is very interesting-to-a-part higher one than algebra.

The theory, Fadeeva, which is a very interesting one, a part of a higher one, i.e., of an algebra.

The Fadeevan theory, which is very interesting to a part higher than algebra.

The Fadeevan one of the theory, who is a very interesting higher-than-a-part algebra.

The Fadeeva of the theory, who is an algebra of a higher one of a part of a very interesting one.

The theory of Fadeeva, which is a very interesting one of a part higher than algebra.

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An Extension of Jensen's Theorem for the Derivative of a Polynomial and for Infrapolynomials

O. Shisha

(February 20, 1962)

The purpose of the paper is to generalize Jensen's theorem on the zeros of the derivative of a real polynomial. Results are first established for infrapolynomials and therefrom derived for expressions of the form $G'(z)$ and $aG(z)+bzG'(z)$, where $G(z)$ is a (complex) polynomial and a,b are (complex) constants. An implicit aim the paper tries to serve is to further show how investigation of infrapolynomials may be of help to the classical study of the geometrical relation between the zeros of a polynomial and those of its derivative (or related polynomials).

1. Let S be a subset of the (open) complex plane. An *infrapolynomial*[1] on S is a polynomial $A(z)\equiv z^n+a_{n-1}z^{n-1}+\ \ldots\ +a_0 (n\geq 1)$ having the property: there does not exist a polynomial $B(z)\equiv z^n+b_{n-1}z^{n-1}+\ \ldots\ +b_0 (\not\equiv A(z))$ such that

$$|B(z)|<|A(z)| \text{ whenever } z\in S \text{ and } A(z)\neq 0, \quad (1)$$

$$B(z)=0 \text{ whenever } z\in S \text{ and } A(z)=0. \quad (2)$$

For example [4], if S is infinite, closed and bounded, then for $n=1, 2, \ldots$ the Tchebycheff polynomial[2] of degree n for S is an infrapolynomial on S.

2. Another example of an infrapolynomial is the following [10, sec. 1, and 2, sec. 10, p. 100. Cf. also 4, sec. 5]. *Let $G(z)$ be a nonconstant polynomial, $\zeta_1, \zeta_2, \ldots, \zeta_m$ its distinct zeros (say $G(z)\equiv\alpha\Pi_{\nu=1}^{m}(z-\zeta_\nu)^{p_\nu}$), and assume that $m\geq 2$. Then*

$$A(z)\equiv\left(\sum_{\nu=1}^{m}p_\nu\right)^{-1}\frac{G'(z)}{G(z)}\ \Pi_{\nu=1}^{m}(z-\zeta_\nu)$$

is an infrapolynomial on $[\zeta_1,\zeta_2, \ldots, \zeta_m]$.

3. Various results have been obtained concerning the location of the zeros of infrapolynomials. The first of these was given (implicitly) by L. Fejér [1] and others were proved by M. Fekete and J. von Neumann [4], J. L. Walsh [27], J. L. Walsh and T. S. Motzkin [31, 11, 12, 13, 14], M. Fekete [2], M. Marden [9], and O. Shisha [16, 17]. One of the most typical theorems in this direction is the following (a special case of [5], Theorem X and the end of sec. 10, p. 66. Cf. also [4], Theorem 1). *Let ξ and η be distinct zeros of an infrapolynomial on a closed and bounded set S. Let L be the perpendicular bisector of the segment (ξ,η). For every z, let C_z denote the closed disk having the segment joining z to its mirror image in L as a diameter ($C_z=[z]$ if $z\in L$), and let C_z' be the closure of the complement of C_z. Then there exists a point p of S such that ξ and η belong to C_p. There exists also a point q of S such that ξ and η belong to C_q'.*

4. In Theorem 1 we generalize the last result by substituting in the latter a circle for the line L.

THEOREM 1: *Let ξ and η be distinct zeros of $A(z)$, an infrapolynomial on a closed and bounded set S. Let K be a circle whose center ζ does not belong to S, such that ξ and η are symmetric (i.e., inverse) to each other with respect to K. For every $z(\neq\zeta)$ let C_z denote the closed disk having the segment joining z to its inverse with respect to K as a diameter ($C_z=[z]$ if $z\in K$), and let C_z' be the closure of the complement of C_z. Then there exists a point p of S such that ξ and η belong to C_p. There exists also a point q of S such that ξ and η belong to C_q'.*

5. To prove this theorem, we establish first the following

LEMMA: *Let $K:|z-\zeta|=r$ be a circle and z_0, ξ_1, ξ_2, η_1, η_2 points such that $z_0\neq\zeta, 0<|\xi_i-\zeta|<r, \eta_i$ is symmetric to ξ_i with respect to $K (i=1, 2)$ and ξ_2 lies on the (open) segment (ξ_1,η_1). Then (using notations of Theorem 1):*

(a) *If $\xi_2\in C_{z_0}'$, then*

$$|z_0-\xi_2|\ |z_0-\eta_2|<|z_0-\xi_1|\ |z_0-\eta_1|. \quad (3)$$

(b) *If $\xi_1\in C_{z_0}$ then*

$$|z_0-\xi_2|\ |z_0-\eta_2|>|z_0-\xi_1|\ |z_0-\eta_1|.$$

PROOF OF THE LEMMA: We introduce a polar coordinate system in which the pole is ζ and the fixed ray is the one emanating from ζ and passing through z_0. Let (ρ_1,φ), (ρ_2,φ) be, respectively, polar coordinates of ξ_1 and $\xi_2 (0<\rho_1<\rho_2<r)$, and set $a=|z_0-\zeta|$. For every positive x let

$$F(x)=(x^2+a^2-2ax\cos\varphi)\left(\left(\frac{r^2}{x}\right)^2+a^2-2a\,\frac{r^2}{x}\cos\varphi\right). \quad (4)$$

[1] Originally called "extremal polynomial." This concept was introduced by M. Fekete and J. von Neumann [4]. The term "infrapolynomial" is due to Professors T. S. Motzkin and J. L. Walsh [12]. Figures in brackets indicate the literature references at the end of this paper.

[2] This is the unique polynomial $T(z)$ of the form (*) $z^n+c_{n-1}z^{n-1}+\ \ldots\ +c_0$ such that the max $|T(z)|$, z on $S]\leq$max $|P(z)|$, z on $S]$ for every polynomial $P(z)$ of the form (*).

Then $(|z_0-\xi_i|\,|z_0-\eta_i|)^2=F(\rho_i)\,(i=1,2)$. Thus in order to prove the Lemma it suffices to show that if $\rho_1<t<\rho_2$, then $F'(t)<0$ in case $\xi_2\epsilon C''_{z_0}$, and $F'(t)>0$ in case $\xi_1\epsilon C'_{z_0}$.

Let t be an arbitrary point of the open interval (ρ_1,ρ_2) and denote by τ the point with polar coordinates (t,φ). Set

$$D=\left|\tau-\frac{1}{2}\,(z_0+z_0')\right|^2-\left|\frac{1}{2}\,(z_0-z_0')\right|^2 \tag{5}$$

where z_0' is the inverse of z_0 with respect to K. If $\xi_2\epsilon C''_{z_0}$, then τ is an interior point of C''_{z_0} and $D>0$. Similarly, if $\xi_1\epsilon C'_{z_0}$ then τ is an interior point of C'_{z_0} and therefore $D<0$.

We have

$$D=t^2+\left[\frac{1}{2}\left(a+\frac{r^2}{a}\right)\right]^2-t\left(a+\frac{r^2}{a}\right)\cos\varphi$$
$$-\left[\frac{1}{2}\left(a-\frac{r^2}{a}\right)\right]^2=t^2+r^2-\frac{t}{a}\,(a^2+r^2)\cos\varphi. \tag{6}$$

From (4), a straightforward computation yields

$$F'(t)=2a^2t^{-3}(t^4-r^4)-2at^{-2}(t^2-r^2)(a^2+r^2)\cos\varphi,$$

and in view of (6),

$$F'(t)=2a^2t^{-3}(t^2-r^2)D.$$

Thus, if $\xi_2\epsilon C''_{z_0}$, then $F'(t)<0$, and if $\xi_1\epsilon C'_{z_0}$, then $F'(t)>0$.

6. Proof of Theorem 1: If one of the two points ξ,η belongs to some C_z, so does the other. Let ξ_1 be that one of ξ,η lying within K, and let η_1 be the other one. So in order to prove the first conclusion of Theorem 1, it suffices to show that $\xi_1\epsilon C$, where $C=\bigcup_{z\epsilon S} C_z$. Suppose $\xi_1\notin C$. Since C is closed, we can find a point ξ_2 within K and on the open segment (ξ_1,η_1) such that $\xi_2\notin C$. Let η_2 be the inverse of ξ_2 with respect to K. Consider the polynomial

$$B(z)\equiv A(z)[(z-\xi_1)(z-\eta_1)]^{-1}(z-\xi_2)(z-\eta_2).$$

By conclusion (a) of the Lemma, for every $z_0\epsilon S$, (3) holds. Thus (1) and (2) hold, contradicting our hypothesis that $A(z)$ is an infrapolynomial on S. Similarly one derives the second conclusion of Theorem 1 from (b) of the Lemma.

7. The notion of an infrapolynomial was generalized by J. L. Walsh and M. Zedek [34] to include more general polynomials, and further work in that direction was done by J. L. Walsh [27, 28, 29], M. Fekete and J. L. Walsh [5, 6], M. Zedek [36, 37, 38], O. Shisha and J. L. Walsh [23], and O. Shisha [16, 17, 20, 21, 22]. (Other papers related to the subject are [3, 10, 11, 12, 15, 18, 19, 31, 32, 33, 35]). A special case studied in much detail by J. L. Walsh [27] is obtained by adding to the properties of $B(z)$ in sec. 1 the equality $b_0=a_0$.

Repeating here a particular instance of a general definition, we mention the following convention. Let $n\,(\geq 2)$ be an integer and S a set in the (open) complex plane. An n-th infrapolynomial on S with respect to $(0,n)$ is a polynomial $A(z)\equiv\Sigma_{\nu=0}^{n}a_\nu z^\nu$ having the property: there does not exist a polynomial $B(z)\equiv\Sigma_{\nu=0}^{n}b_\nu z^\nu\,(\not\equiv A(z)$ satisfying (1), (2) and the equalities $b_0=a_0$, $b_n=a_n$.

8. Corresponding to sec. 2, we have the following result which is a special case of a more general theorem [23, Theorem 3]. *Let $G(z)$ be a polynomial $(G(0)\neq 0)$, $\zeta_1,\zeta_2,\ldots,\zeta_m\,(m\geq 2)$ its distinct zeros, and let* a *and* b *be arbitrary complex numbers. Then*

$$A(z)\equiv\left[a+bz\,\frac{G'(z)}{G(z)}\right]\Pi_{\nu=1}^{m}(z-\zeta_\nu) \tag{7}$$

is an m-th infrapolynomial on $[\zeta_1,\zeta_2,\ldots,\zeta_m]$ with respect to $(0,m)$.

9. Theorem 2: *Theorem 1 remains true if "infrapolynomial" is replaced by "n-th$(n\geq 2)$ infrapolynomial $(\not\equiv 0)$ with respect to $(0,n)$," provided that the center ζ of K is the origin.*

Indeed, we can use again the proof of Theorem 1 since $B(z)$ defined there will have the same coefficient of z^n as $A(z)$ has, and (since $|\xi_2\eta_2|=$square of the radius of $K=|\xi_1\eta_1|$, which implies that $\xi_2\eta_2=\xi_1\eta_1$) the constant terms of $B(z)$ and $A(z)$ will be equal.

10. As an application of theorems 1 and 2 we have the following

Theorem 3: I. *Let ξ and η be distinct zeros of $G'(z)$ where $G(z)$ is a nonconstant polynomial. Let ξ and η be symmetric with respect to a circle K whose center is not a zero of $G(z)$. Then there exists a zero p of $G(z)$ such that (using a notation of Theorem 1) ξ and η belong to C_p. There exists also a zero q of $G(z)$ such that ξ and η belong to C'_q.* II. *Let ξ and η be distinct zeros of $aG(z)+bzG'(z)\,(\not\equiv 0)$ where $G(z)$ is a polynomial $(G(0)\neq 0)$ and a and b are complex numbers. Let ξ and η be symmetric with respect to a circle K. Then the conclusions of part I hold, provided that the center of K is the origin.*

Proof: We set $G(z)\equiv\alpha\Pi_{\nu=1}^{m}(z-\zeta_\nu)^{p_\nu}$ where $\zeta_j\neq\zeta_k$ whenever $j\neq k$ and where $p_1,p_2,\ldots,p_m$ are positive. To prove part I we define $A(z)$ as in sec. 2 and observe that $m\geq 2$ since $G'(z)$ vanishes at the distinct points ξ and η. We may assume that $G(\xi)G(\eta)\neq 0$, for otherwise we can take $p=q=\xi$ or $p=q=\eta$. Thus ξ and η are zeros of $A(z)$, which by sec. 2 is an infrapolynomial on the set of zeros of $G(z)$. The desired conclusions follow now from Theorem 1. To prove part II, we assume that $m\geq 2$, as otherwise ξ or η is ζ_1 and one can take $p=q=\zeta_1$. Define now $A(z)$ by (7). As before, we may also assume that $G(\xi)G(\eta)\neq 0$. Then ξ and η are zeros of $A(z)$, which by sec. 8 is an m-th infrapolynomial on $[\zeta_1,\zeta_2,\ldots,\zeta_m]$ with respect to $(0,m)$. The desired conclusions follow from Theorem 2.

11. For every z, let Γ_z and Γ'_z be, respectively, C_z and C'_z of sec. 3, with L taken as the real axis. Theorem 3, I generalizes the following well known theorem, enunciated without proof by J. L. W. V.

Jensen [7] and proved by J. L. Walsh [24]. *Let ξ be a nonreal zero of $A'(z)$, where $A(z)$ is a nonconstant polynomial whose coefficients are real. Then there exists a zero p of $A(z)$ such that $\xi \epsilon \Gamma_p$.* It was proved also by J. L. Walsh that there exists a zero q of $A(z)$ such that $\xi \epsilon \Gamma'_q$.

12. The last results readily imply the following limiting case of Theorem 3, I. *Let ξ and η be distinct zeros of $G'(z)$, where $G(z)$ is a nonconstant polynomial. Let L be the perpendicular bisector of the segment (ξ,η). Then (using the notation of sec. 3) there exists a zero p of $G(z)$ such that ξ and η belong to C_p. There exists also a zero q of $G(z)$ such that ξ and η belong to C''_q.* Indeed, we may assume (using a simple transformation) that $\eta = \overline{\xi}$. Set $G(z) \equiv \alpha \Pi_{\nu=1}^{n}(z - z_\nu)$, $A(z) \equiv \Pi_{\nu=1}^{n}(z - z_\nu)(z - \overline{z}_\nu)$. Then $A'(\xi) = (\Pi_{\nu=1}^{n}(z - z_\nu))'_{z=\xi} \Pi_{\nu=1}^{n}(\xi - \overline{z}_\nu) + \Pi_{\nu=1}^{n}(\xi - z_\nu)(\Pi_{\nu=1}^{n}(z - z_\nu))'_{z=\eta} = 0$. The desired conclusions follow now[3] from sec. 11. These same conclusions are implied also by the results quoted in secs. 2, 3.

13. There are many complements and generalizations of Jensen's theorem due to J. L. Walsh for which the reader is referred to the literature [24, 25, 26, 27, 30].

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[3] Compare with [5], sec. 16, p. 66.

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Two Matrix Eigenvalue Inequalities

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A lower bound is given for the quantity λ_1/λ_n, and an upper bound for the quantity $\lambda_1-\lambda_n$, where λ_1 and λ_n are respectively the greatest and least characteristic roots of a matrix with positive roots. The bounds involve the first and second coefficients of the characteristic equation of the matrix.

Suppose $A=(a_{ij})$ is a nonsingular $n\times n$ matrix with characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, so ordered that $|\lambda_1|\geq|\lambda_2|\geq\ldots\geq|\lambda_n|$. The quantity $|\lambda_1|/|\lambda_n|$ provides a rough measure of the probable error in the computation of the inverse of A; it has been called by J. Todd [1, 2, 3][1] the P-condition number of A and may be denoted $P(A)$. Von Neumann and Goldstine [4] have shown that if A is symmetric and positive definite (in which case the λ_i are all positive), then the error in the inverse of A computed by a certain elimination method, can be bounded by a quantity proportional to $P(A)$; if A is not symmetric positive definite, the error can be bounded by a quantity proportional to $P(AA')$.

We shall restrict our consideration to matrices whose roots are all positive. For these, P. J. Davis, E. V. Haynsworth, and M. Marcus [5] obtained bounds on P involving $\det A$ and one other symmetric function of the roots of A. If the characteristic polynomial of A is $p(x)=x^n-C_1x^{n-1}+C_2x^{n-2}+\ldots+(-1)^nC_n$ and we set $D_1=(n^nC_n)/C_1^n$, they showed that

$$\frac{1}{D_1^{n-1}}\leq P\leq\frac{1+\sqrt{1-D_1}}{1-\sqrt{1-D_1}}; \tag{1}$$

and they found similar inequalities involving C_n ($=\det A$) and any other one of the C_i.

In many cases, however, C_n is not known and it is in general difficult to calculate. C_1($=$trace A) is easy to calculate, and C_2 can in general be calculated more easily than can C_n since it is the sum of $[n(n-1)]/2$ determinants of order two. In this paper we present (in Theorems 1 and 1′) a lower bound for P in terms of C_1 and C_2; an attempt to obtain a corresponding upper bound fails, but leads to an inequality (Theorem 2) on $\lambda_1-\lambda_n$, the "spread" of the roots of A. Finally we apply the method of proof of Theorem 1 to obtain an improvement of the lower bound in (1).

[1] Figures in brackets indicate the literature references at the end of this paper.

If we set $C_K=\binom{n}{K}S_K(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $S_K^{1/K}$ is the K'th symmetric mean of $\lambda_1, \ldots, \lambda_n$. The S_K satisfy [6] the inequalities

$$S_1\geq S_2^{1/2}\geq S_3^{1/3}\geq\ldots\geq S_n^{1/n}. \tag{2}$$

Setting $\mu_K=\frac{\lambda_K}{\lambda_n}$, we have $P=\mu_1\geq\mu_2\geq\mu_3\geq\ldots\geq\mu_n=1$,

and

$$\frac{S_1^2(\mu_1, \mu_2, \ldots, \mu_n)}{S_2(\mu_1, \mu_2, \ldots, \mu_n)}=\frac{S_1^2(\lambda_1, \ldots, \lambda_n)}{S_2(\lambda_1, \ldots, \lambda_n)}.$$

Let

$$R(x_2, x_3, \ldots, x_{n-1})=\frac{S_1^2(P, x_2, x_3, \ldots, x_{n-1}, 1)}{S_2(P, x_2, x_3, \ldots, x_{n-1}, 1)}$$

and

$$f_1(P)=\max_{1\leq x_i\leq P} R(x_2, x_3, \ldots, x_{n-1})$$

and

$$f_2(P)=\min_{1\leq x_i\leq P} R(x_2, \ldots, x_{n-1}).$$

Then f_1 and f_2 can be seen to be increasing in P, and

$$f_2(P)\leq\frac{S_1^2}{S_2}\leq f_1(P). \tag{3}$$

Thus the right half of (3) should provide a lower bound for P, while the left half should provide an upper bound. We first calculate $f_2(P)$:

By direct calculation we can show that $(\partial^2R)/\partial x_i^2$ is nonnegative at all points for each i. Therefore R attains its maximum at a point where each x_i is either 1 or P. Letting $R_K(P)$ denote the value of R when $K-1$ of $x_2, x_3, \ldots, x_{n-1}$ are equal to P and the remainder are equal to one, we find that R_K is equal to:

$$\frac{n-1}{n}\frac{(KP+n-K)^2}{(KP+n-K)^2-(KP^2+n-K)}. \tag{4}$$

This rational function of K attains its maximum at $K=\frac{n}{P+1}$ and so we obtain

$$\frac{S_1^2}{S_2}\leq\frac{n-1}{n}\,\frac{1}{1-\dfrac{(P+1)^2}{4Pn}} \tag{5}$$

which is equivalent to:

THEOREM 1:

$$\frac{1+\sqrt{1-A}}{1-\sqrt{1-A}}\leq P, \text{ where } A=\frac{1}{n-\dfrac{S_2}{S_1^2}(n-1)}.$$

The upper bound (5) can be sharpened, since in fact we need only consider integer values of K in (4). It can be shown that $R_1=\max_K R_K$ if $P^2\geq[(n-1)(n-2)]/2$. Thus if $P^2\geq[(n-1)(n-2)]/2$, $S_1^2/S_2\leq R_1(P)$; and so, setting $\rho=S_1^2/S_2$, we have:

$$[(\rho-1)+\sqrt{\rho(\rho-1)}]\,n+1\leq P. \tag{6}$$

Now if $P^2<[(n-1)(n-2)]/2$, then $\rho\leq\max_K R_K(P)=R_K{}^*(P)$, say; and since each $R_K(P)$ is a strictly increasing function of P, $\rho<R_K{}^*([(n-1)(n-2)]/2)\leq R_1([(n-1)(n-2)]/2)$ and so $\sqrt{\frac{(n-1)(n-2)}{2}}>R_1^{-1}(\rho)$. However it can easily be seen that if $\rho\geq(\sqrt{2}+\frac{3}{2})/(\sqrt{2}+1)$, then $R_1^{-1}(\rho)\geq\sqrt{\frac{(n-1)(n-2)}{2}}$; and so we conclude that:

THEOREM 1′: *If* $\rho\geq(\sqrt{2}+\frac{3}{2})/(\sqrt{2}+1)$, *then* $[(\rho-1)+\sqrt{\rho(\rho-1)}]\,n+1\leq P$.

This lower bound is better than the previous one, whenever it applies.

The attempt to derive an upper bound for P from the left half of (3) fails, because $f_2(P)$ approaches 1 uniformly in P as n increases, and so the inequality $f_2(P)\leq S_1^2/S_2$ will in most cases hold for all values of P. We can, however, by considering the minimum of $S_1^2(\lambda_1, y_2, y_3, \ldots, y_{n-1}, \lambda_n)-S_2(\lambda_1, y_2, \ldots, y_{n-1}, \lambda_n)$ subject to the condition $\lambda_1\geq y_2\geq y_3\geq\ldots\geq y_{n-1}\geq\lambda_n$, obtain an upper bound for the spread of the roots of A. Calling the above function of $y_2, \ldots, y_{n-1}$ D, we evaluate its minimum directly by observing that $\partial D/\partial y_i=-2/[n(n-1)](S_1-y_i)$ and $\partial^2 D/\partial y_i\partial y_j=2/n^2$ if $i=j$ and $-2/[n^2(n-1)]$ if $i\neq j$; $i, j=2, 3, \ldots, n-1$. The $(n-2)\times(n-2)$ matrix (d_{ij}), with $d_{ij}=\partial^2 D/\partial y_i\partial y_j$ is symmetric, and by Gerschgorin's theorem each of its eigenvalues lies in the circle $|Z-2/n^2|\leq 2/n^2\left(\frac{n-3}{n-1}\right)$ and so is positive. Thus (d_{ij}) is positive definite, and, setting each $\partial D/\partial y_i$ equal to zero, we find that D is at a minimum when each y_i is equal to $(\lambda_1+\lambda_n)/2$, and that the minimal value is $[(\lambda_1-\lambda_n)^2]/[2n(n-1)]$. We may then conclude:

THEOREM 2: $\lambda_1-\lambda_n\leq\sqrt{2n(n-1)(S_1^2-S_2)}$.

By the method of Theorem 1, it is possible to sharpen the left side of inequality (1). Following the notation of [5] we write

$$D_1(x_1, x_2, \ldots, x_n)=\frac{[S_n(x_1, \ldots, x_n)]}{[S_1^n(x_1, \ldots, x_n)]}=\frac{x_1\times x_2\times\ldots\times x_n}{\left(\dfrac{x_1+\ldots+x_n}{n}\right)^n},$$

and seek an upper bound for D_1 subject to the condition $P=x_1\geq x_2\geq\ldots\geq x_n=1$. As in the proof of Theorem 1, we show that $\partial^2 D_1/\partial x_i^2\geq 0$ for $i=2, 3, \ldots, n-1$ and finally obtain the relation

$$D_1\leq\left[\frac{P-1}{\log P}\,e^{-1+\frac{\log P}{P-1}}\right]^n \tag{7}$$

which leads to the inequality:

THEOREM 3:

$$\frac{e}{D_1^{\frac{1}{n}}}\leq\frac{P-1}{\log P}\,e^{\frac{\log P}{P-1}}.$$

This inequality yields a lower bound for P which is always higher than that given by (1) (as may be seen by comparing the proof of Theorem 3 with the proof of (1) in [5]). However it is cumbersome. It can be simplified (and somewhat weakened) as follows: Since $xe^{1/x}=a$, $a\geq e$, implies that $x\geq a-e+1$, and $e/D_1^{\frac{1}{n}}\geq e$ by (2), we may conclude that $(P-1)/\log P\geq e/D_1^{\frac{1}{n}}-e+1$. Denoting this last quantity by A, we have $(P-1)/\log P\geq A$, which has as an immediate consequence:

THEOREM 3′: $P\geq A\log A-1$; $A=e/D_1^{\frac{1}{n}}-e+1$.

This lower bound is most often, though not always, better than that given in (1).

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(Paper 66B2–73)

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 66B, No. 2, April-June 1962

Graphs for Determining the Power of Student's *t*-Test

Mary C. Croarkin

(January 9, 1962)

This paper presents charts for determining the operating characteristics of Student's *t*-test. For a fixed level of significance α, the charts give constant contours of the power β depicting the number of observations plotted against a function of the difference in means. Both the one-sample and the two-sample (equal sample sizes) cases are treated for all combinations of

α=0.01, 0.02, 0.05, 0.10, 0.20 and
β=0.10, 0.50, 0.90, 0.95, 0.99 except
(α,β)=(0.10, 0.10), (0.10, 0.20).

The graphs were constructed for two-sided tests but within the given accuracy are equally applicable to one-sided tests.

1. Introduction

It is the purpose of this paper to provide charts for readily ascertaining the operating characteristics or power curve of Student's *t*-test. When a statistical test procedure is being selected, oftentimes the power of the test is not examined because of the difficulties in computing the power curve, or at best the power is computed at a single point. The charts given in this paper will provide four or five (depending on the level of significance) points on the power curve, and these points together with the level of significance should be sufficient for a graphical approximation of the power function.

Other tabulations of the noncentral *t*-distribution have been made by Neyman [3],[1] Neyman and Tokarska [4], Resnikoff [5], Resnikoff and Lieberman [6], Johnson and Welch [2], and Davies (Tables E and E.1) [1].

The tables given in Davies [1] are by far the most convenient for evaluating the power of Student's *t*-test in routine applications; the charts presented here are essentially a graphical extension of these tables. All charts are the result of calculations done at the National Bureau of Standards in connection with a table of the noncentral *F* distribution [7].

2. One-Sample Case

Assume $X_1, \ldots, X_n$ is a random sample of n observations from a normal distribution with mean μ and variance σ^2 and that we are interested in testing the null hypothesis H_0 that $\mu=\mu_0$ against the alternate hypothesis H_1 that $\mu\neq\mu_0$. If H_0 is true, then the statistic [2]

$$t=\frac{\sqrt{n}(\overline{X}-\mu_0)}{s_x} \tag{1}$$

is known to follow Student's *t*-distribution with $(n-1)$ degrees of freedom; however, if H_1 is true, t follows a *noncentral t* distribution with $(n-1)$ degrees of freedom.

The usual test procedure is to choose the level of significance α from which a value t_α can be determined such that

$$Pr\{|t|>t_\alpha|H_0 \text{ true}\}=\alpha. \tag{2}$$

Then the proper decision rule at the α level of significance is to

$$\begin{aligned}&\text{accept } H_0 \text{ if } |t|\leq t_\alpha\\ &\text{accept } H_1 \text{ if } |t|>t_\alpha.\end{aligned} \tag{3}$$

When the alternate hypothesis H_1 is true, i.e., when $\mu\neq\mu_0$, then the probability of accepting H_1 is termed the *power of the test* and is given by

$$\beta=Pr\{|t|>t_\alpha|H_1 \text{ true}\}=Pr\{\text{accept } H_1|H_1 \text{ true}\}. \tag{4}$$

3. Two-Sample Case

Suppose we are given two random samples from normal populations, say, $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ with means μ_1 and μ_2 respectively and unknown variances σ^2. Then the problem of testing the null hypothesis H_0 that $\mu_1=\mu_2$ against the alternate hypothesis H_1 that $\mu_1\neq\mu_2$ is resolved by considering the *t*-statistic [3]

$$t=\frac{\sqrt{n}(\overline{X}-\overline{Y})}{\sqrt{s_x^2+s_y^2}} \tag{5}$$

with $2(n-1)$ degrees of freedom and applying the test procedure indicated by (3).

[1] Figures in brackets indicate the literature references at the end of this paper.

[2] $\overline{X}=\sum_1^n X_i/n$, $s_x=\sqrt{\Sigma(X_i-\overline{X})^2/(n-1)}$.

[3] $\overline{Y}=\sum_1^n Y_i/n$, $s_y=\sqrt{\Sigma(Y_i-\overline{Y})^2/(n-1)}$.

4. Use of Charts

The power of Student's t-test depends upon three parameters: Δ, the difference between means where $\Delta=|\mu-\mu_0|$ for the one-sample case, and $\Delta=|\mu_1-\mu_2|$ for the two-sample case; the number of observations; and the level of significance. The charts given in this paper depict a standardized difference $\delta=\Delta/\sigma$ plotted against n for fixed α and β. Figures 1–5 deal with the one-sample case; whereas figures 6–10 deal with the two-sample case. It should be noted that on all graphs dealing with the two-sample case, n is the number of observations in *each* sample.

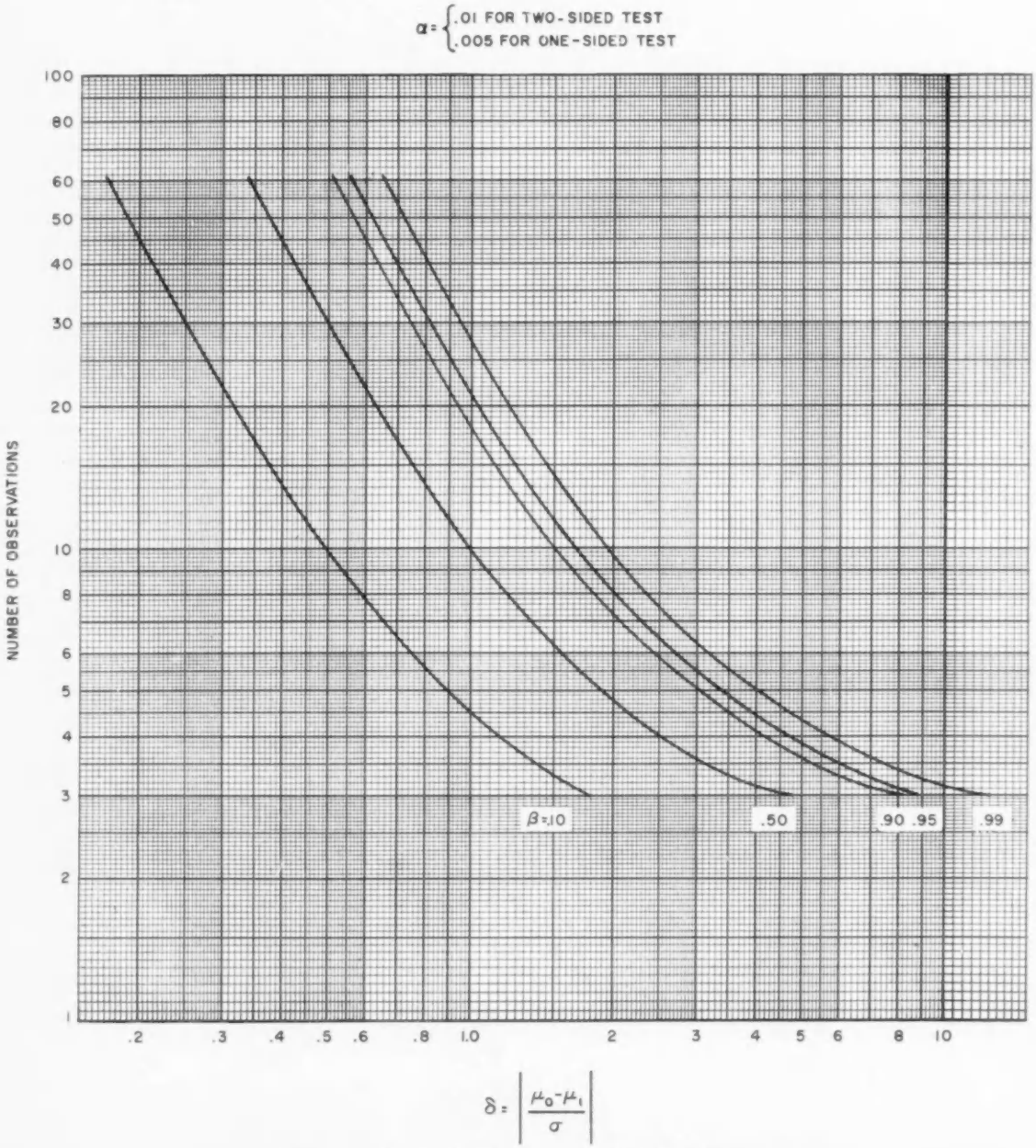

FIGURE 1. *Curves of constant power (β) for t-test of mean.*

All charts were constructed for a two-sided or symmetrical t-test at level of significance α. However, within the accuracy of these charts, the one-sided test will be equivalent to the two-sided test, and hence these same graphs can be used for an asymmetrical t-test where the appropriate level of significance will now be $\alpha/2$ instead of α.

5. Applications

Example 1. Suppose that we plan to take $n=10$ observations and carry out a two-sided t-test for the population mean at level of significance $\alpha=0.05$ and that we are interested in the power curve of such a test.

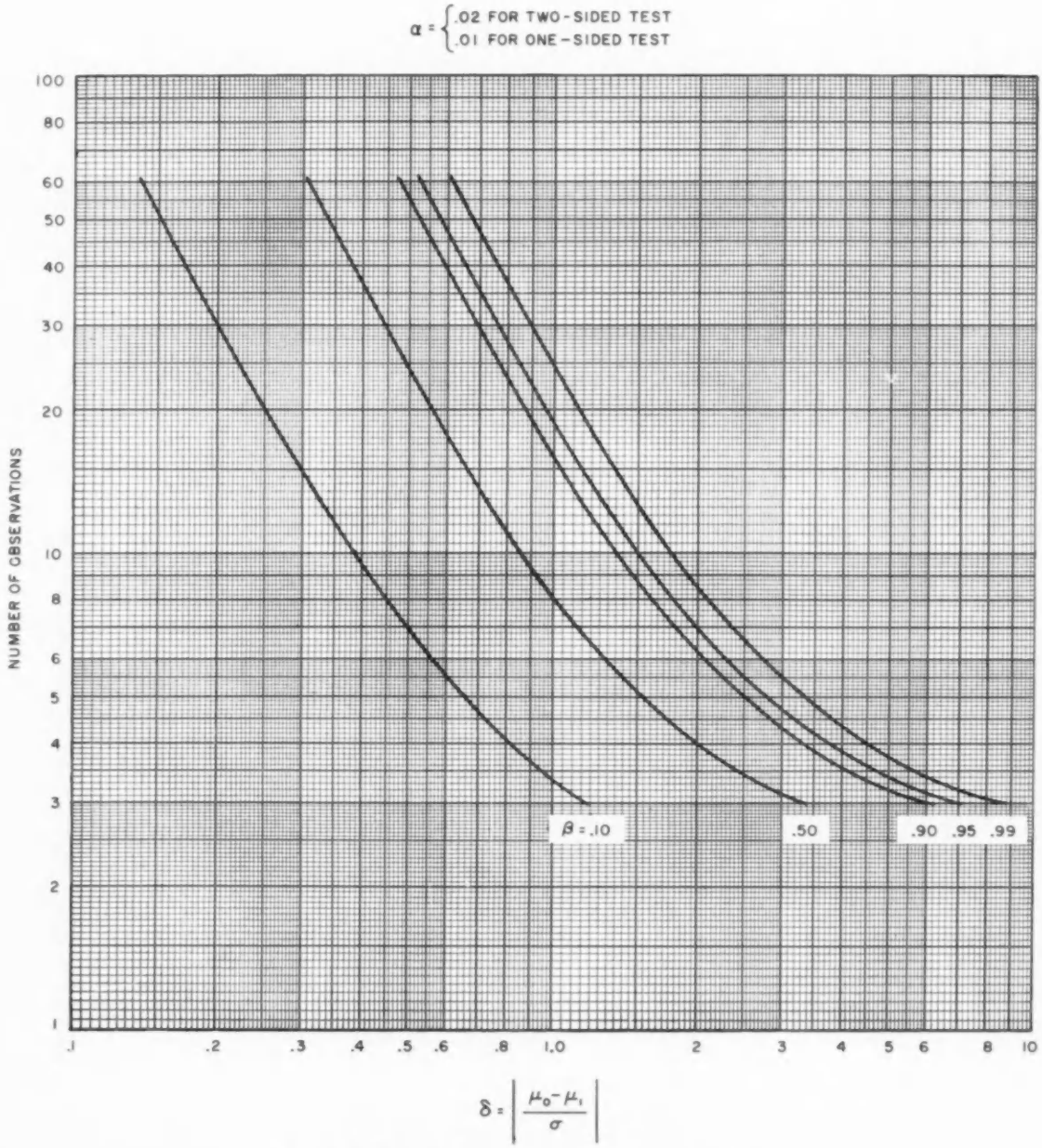

FIGURE 2. *Curves of constant power (β) for t-test of mean.*

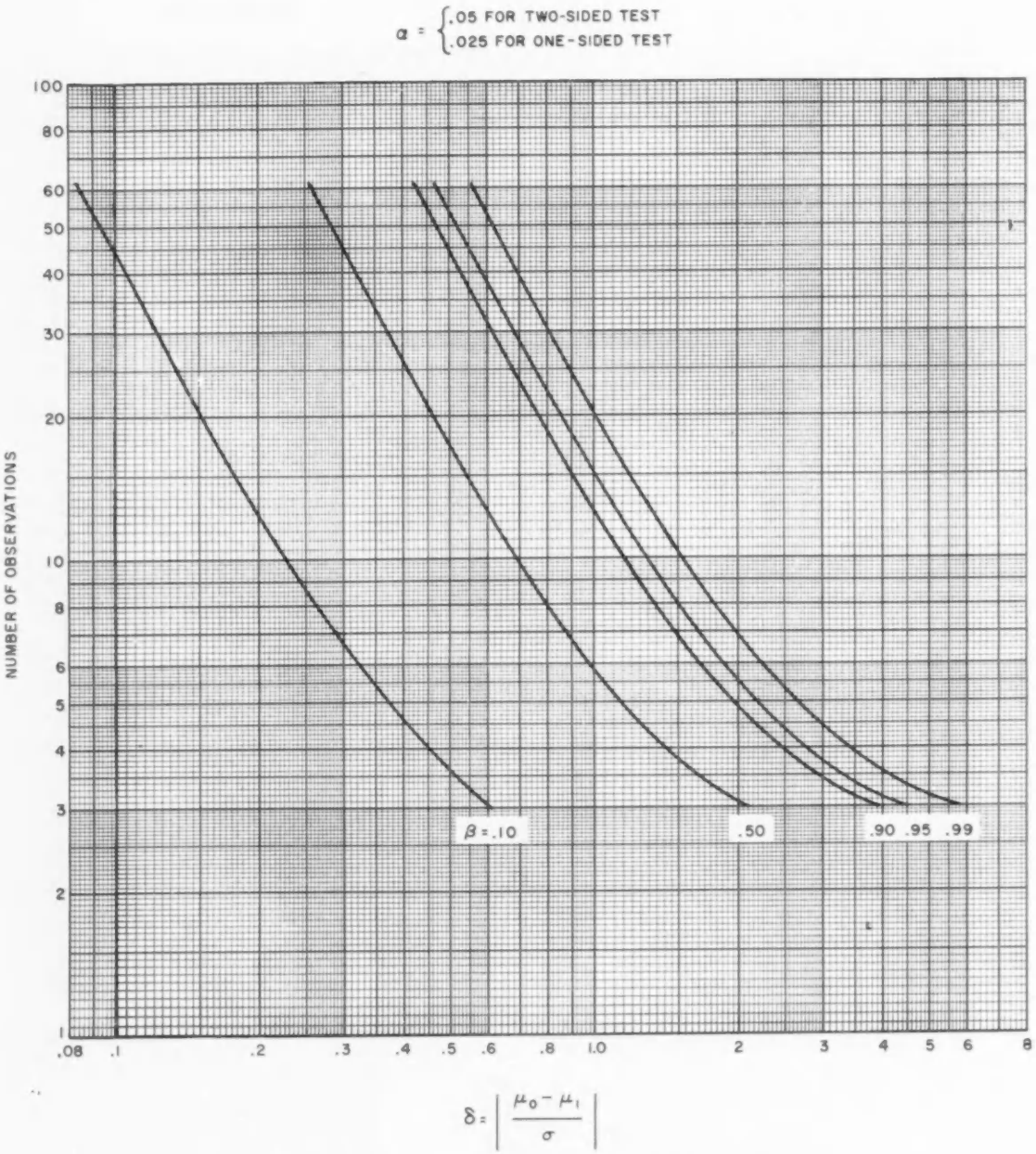

FIGURE 3. *Curves of constant power (β) for t-test of mean.*

Locating the horizontal line at $n=10$ on figure 3, we can read off the five values of the standardized difference $\delta=\frac{|\mu-\mu_0|}{\sigma}$ which can be detected with probability $\beta=0.10$, 0.50, 0.90, 0.95, and 0.99. These five points, which are tabulated below, together with the level of significance $\alpha=0.05$ are sufficient to graph the power curve of that particular test. A sketch of this graph is given in figure 11.

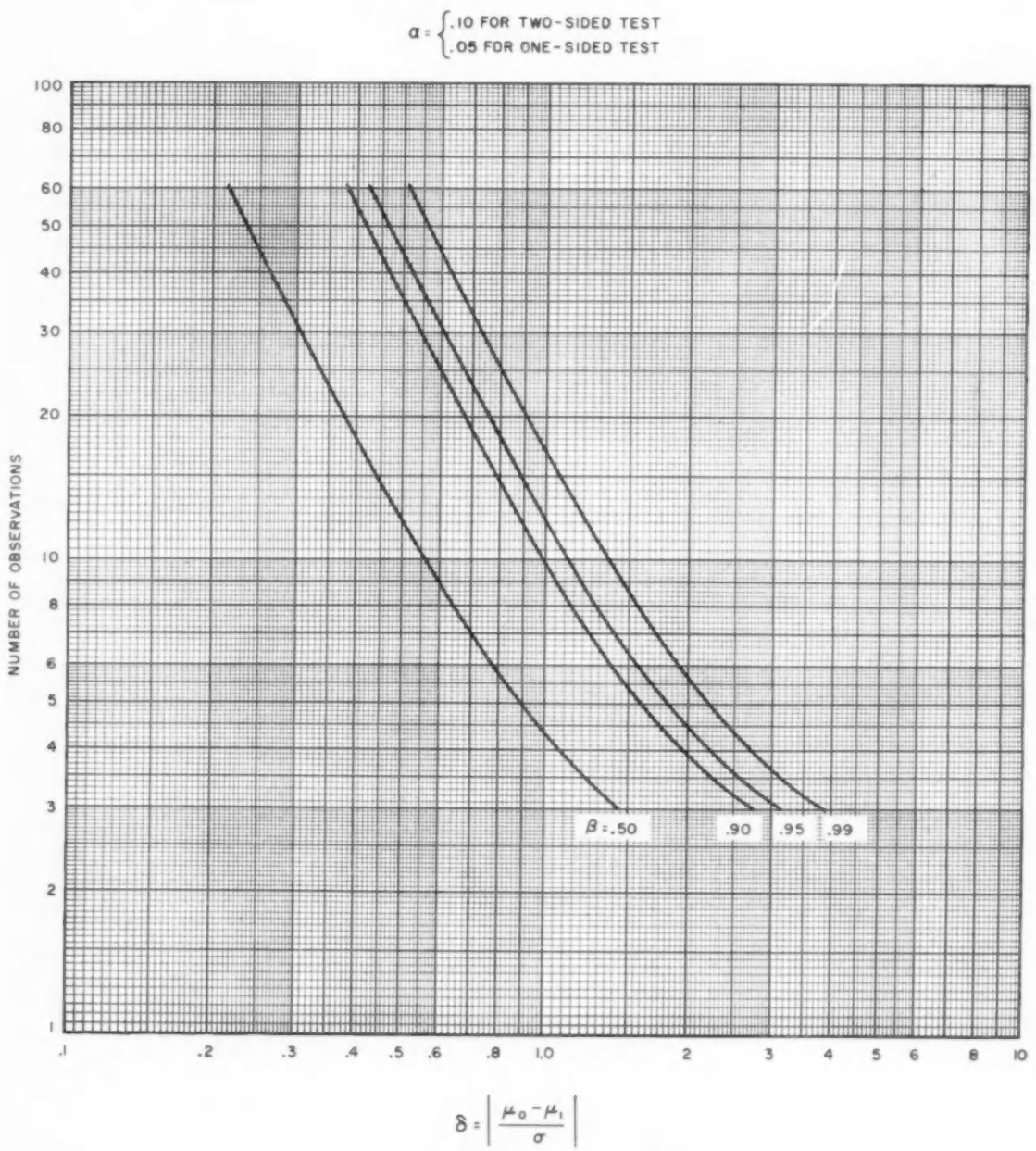

FIGURE 4. *Curves of constant power (β) for t-test of mean.*

Power (β)	$\frac{\lvert\mu-\mu_0\rvert}{\sigma}$
0.10	0.23
.50	.69
.90	1.15
.95	1.30
.99	1.52

The entire power curve enables one to find the corresponding power β for any standardized difference δ. For example, the probability of detecting a difference of one standard deviation, $\delta = 1.0$, would be found from the graph to be 0.76.

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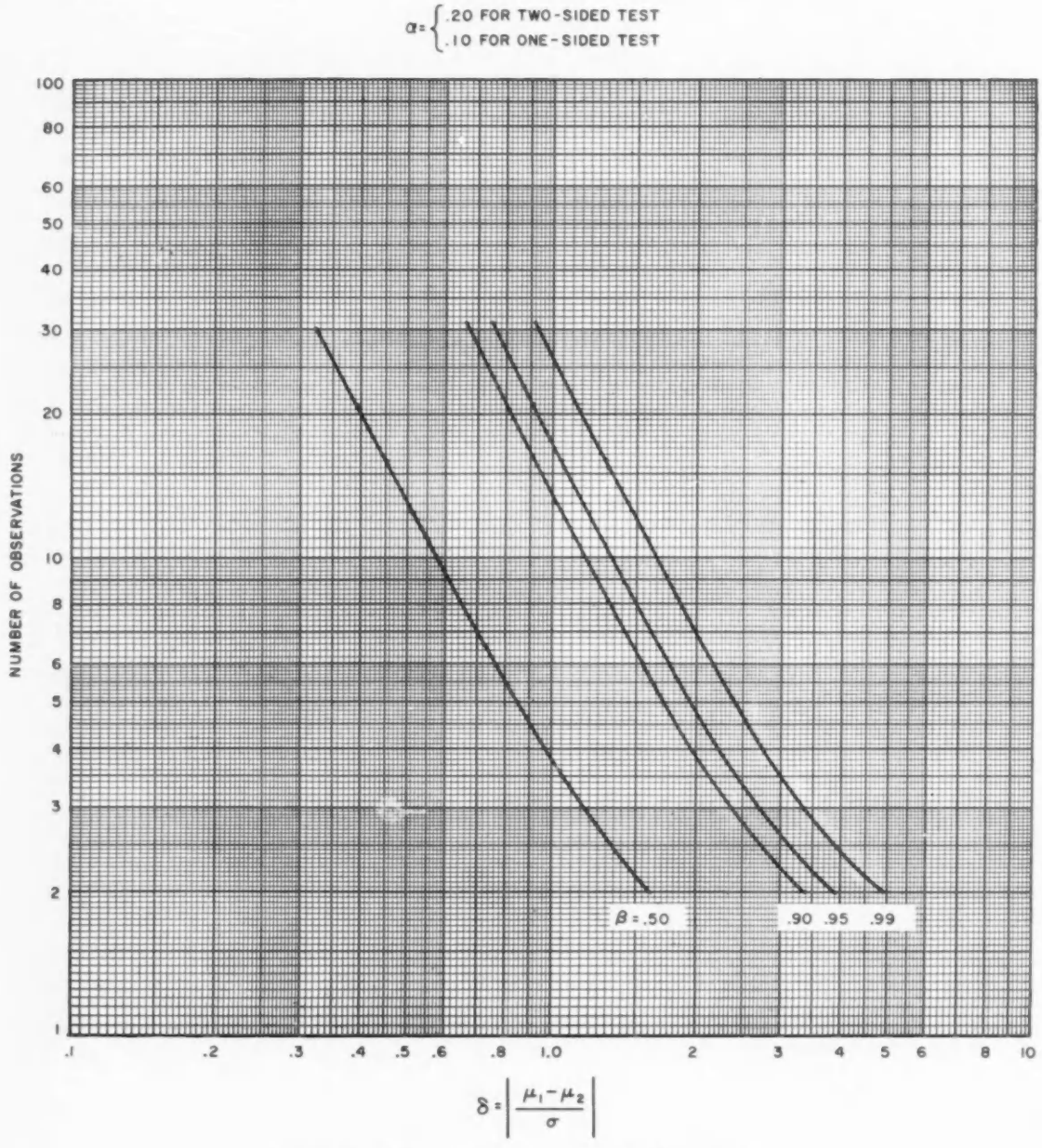

FIGURE 5. *Curves of constant power (β) for* t*-test of mean.*

Example 2. How many observations are necessary to detect a difference of at least 2σ between the means of two normal populations with a probability of $\beta=0.95$ using a two-sided t-test at level of significance $\alpha=0.10$?

Using figure 9 we see that the $\delta=2$ line intersects the $\beta=0.95$ curve at approximately 6.2. Therefore, seven observations from each population would be necessary to detect a difference of 2σ between the population means.

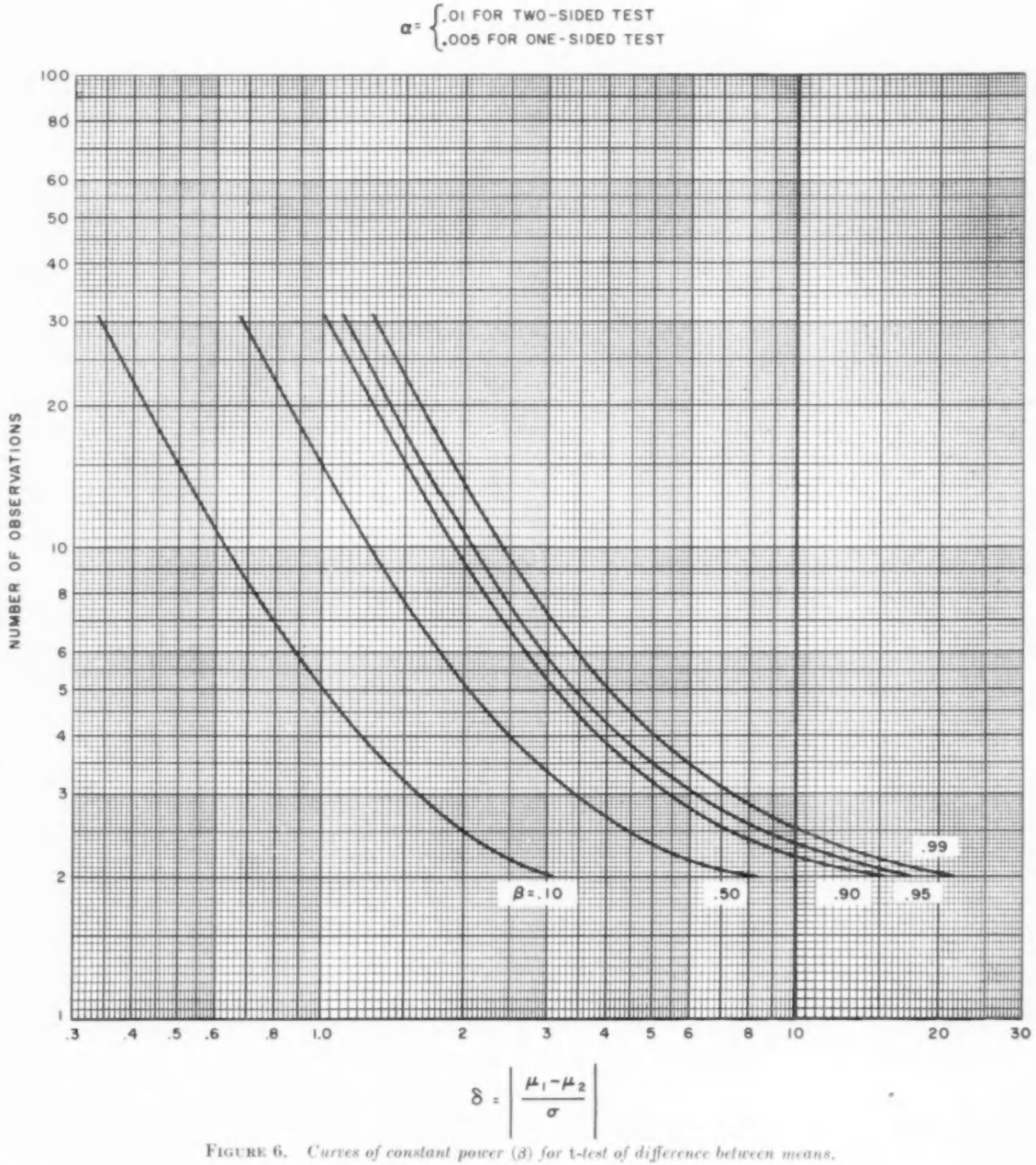

FIGURE 6. *Curves of constant power (β) for t-test of difference between means.*

Example 3. It is planned to carry out a one-sided t-test for comparing two population means at level of significance $\alpha=0.01$. The problem is to find the power curve when $n=5$ observations are taken from each population.

From figure 7 we see that the values of the standardized difference δ corresponding to $\beta=0.10$, 0.50, 0.90, 0.95, 0.99 for $n=5$ are: (see fig. 12)

Power (β)	$\delta=\left\|\frac{\mu_1-\mu_2}{\sigma}\right\|$
0.10	0.78
.50	1.77
.90	2.78
.95	3.08
.99	3.60

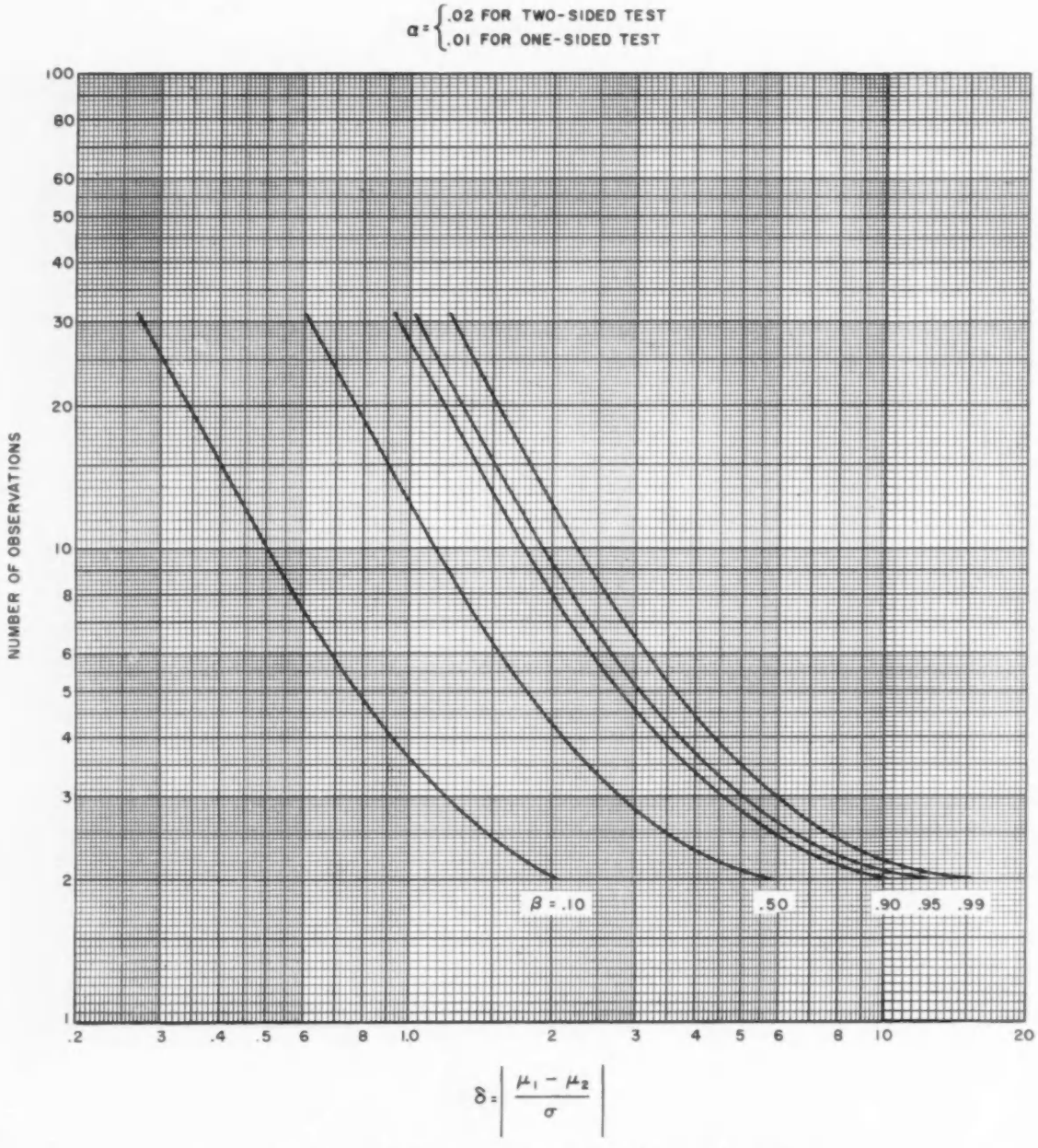

FIGURE 7. *Curves of constant power (β) for t-test of difference between means.*

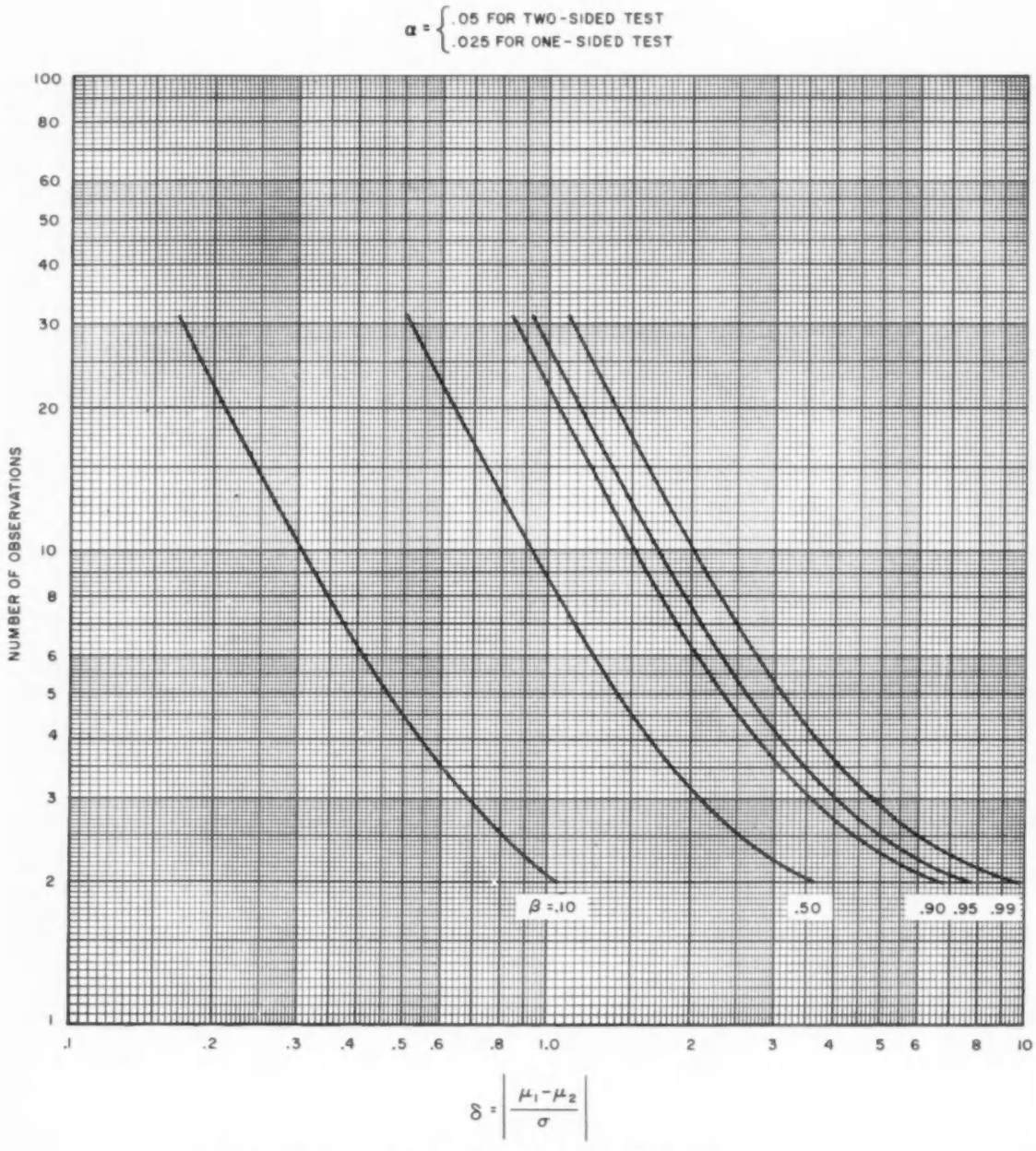

FIGURE 8. *Curves of constant power (β) for t-test of difference between means.*

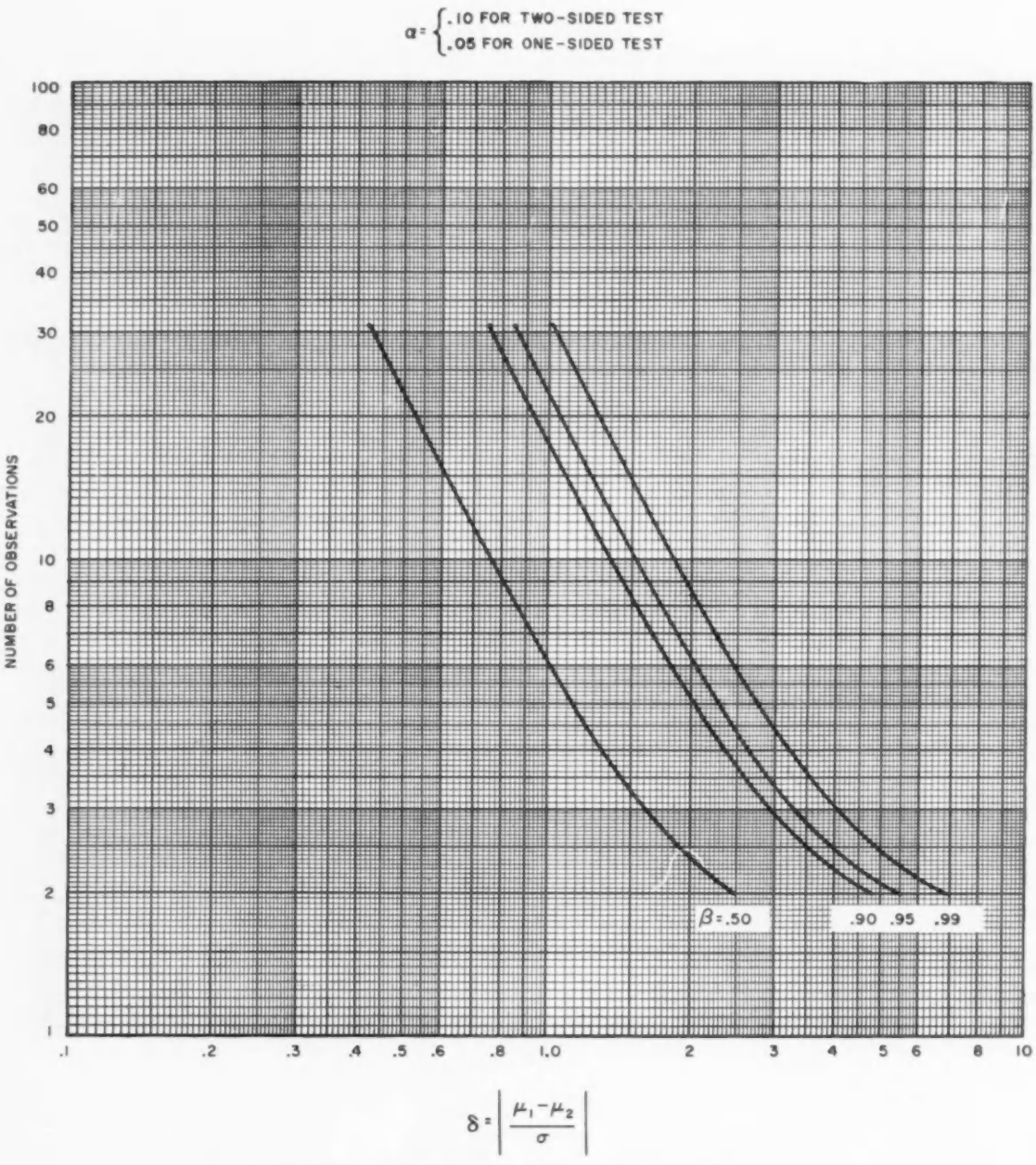

FIGURE 9. *Curves of constant power (β) for t-test of difference between means.*

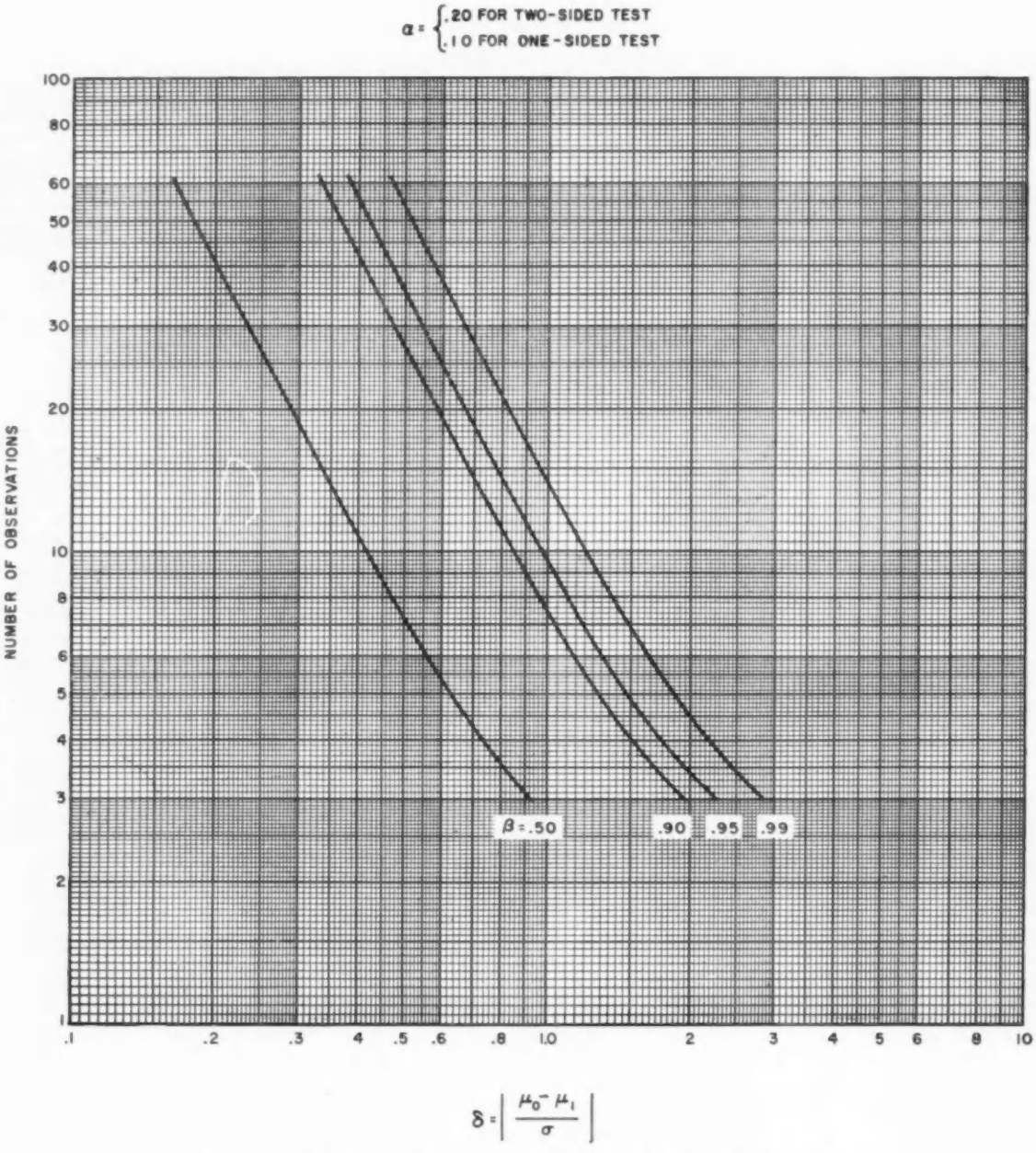

FIGURE 10. *Curves of constant power (β) for t-test of difference between means.*

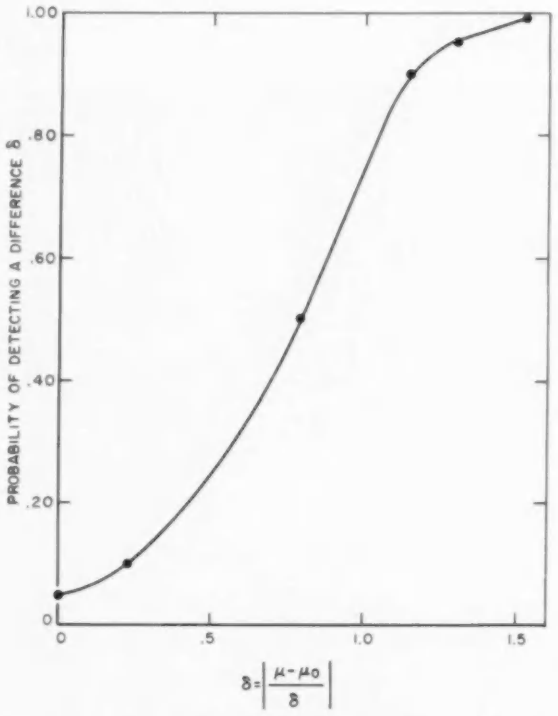

FIGURE 11. *Power curve for a two-sided* t*-test of the population mean where* n=10 *and* α=0.05.

The author thanks Marvin Zelen for innumerable contributions to this paper.

(Paper 66B2–74)

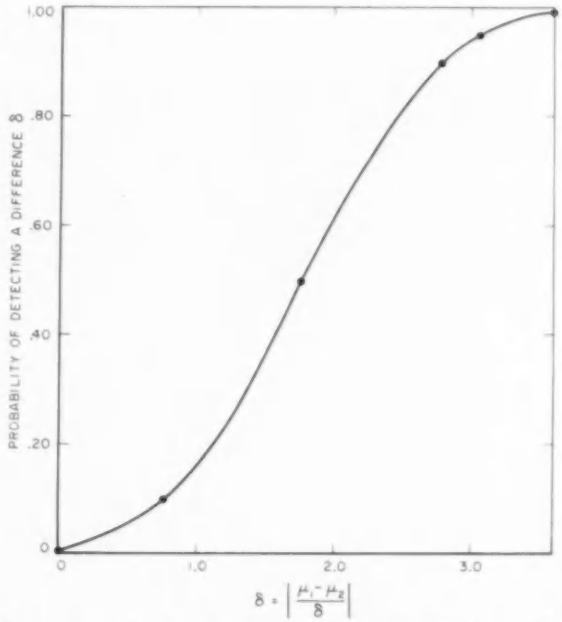

FIGURE 12. *Power curve for a one-sided* t*-test for comparing the means of two populations where* n=5 *and* α=0.01.

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[7] Tables of the non-central *F* distribution, to be published by the National Bureau of Standards as part of its Applied Mathematics Series.

JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 66B, No. 2, April–June 1962

Publications of the National Bureau of Standards

Selected Abstracts

Systems of electrical units, F. B. Silsbee, *J. Research NBS* **66C** (*Engr. and Instr.*) *No. 2* (*Apr.–June 1962*) *75 cents.*
The various systems of measurement, with their respective sets of units, used in the literature on electricity and magnetism are described in detail. Their historical development is summarized. The manner in which each is derived from either of the two alternative points of view of the experimentalist and the theoretician is compared and contrasted. The desirability of recognizing both points of view in international standardization, particularly when discussing rationalization, is pointed out. The present status of the absolute measurements on which all electrical units are based is reported, and tables are included for the conversion of equations and numerical values from one system to another.

Some problems connected with Rayleigh distributions, M. M. Siddiqui, *J. Research NBS* **66D** (*Radio Prop.*) *No. 2* (*Mar.–Apr. 1962*) *70 cents.*
This is an expository paper presenting the following: (1) the origin, and (2) the properties of the Rayleigh distribution; (3) the most efficient estimators of its parameters; (4) a test of the hypothesis that a set of observations is from a Rayleigh distribution; (5) the distribution of the ratio of two independent Rayleigh variates; and (6) the Rayleigh process derived from a normal process.

Impedance of a monopole antenna with a radial-wire ground system on an imperfectly conducting half space. Part I., S. W. Maley and R. J. King, *J. Research NBS* **66D** (*Radio Prop.*) *No. 2* (*Mar.–Apr. 1962*) *70 cents.*
The effectiveness of a radial wire ground system as an approximation to a radial conducting disk ground system for a vertical monopole antenna over an imperfectly conducting ground is investigated experimentally by means of impedance measurements. The results were compared with theoretical work by J. R. Wait. The comparison shows that Wait's formula for the effective surface impedance of a radial wire ground system gives results which agree well with the measurements.

Theory of the infinite cylindrical antenna including the feedpoint singularity in antenna current, R. H. Duncan, *J. Research NBS* **66D** (*Radio Prop.*) *No. 2* (*Apr.–Mar. 1962*) *70 cents.*
The usual idealized model for a cylindrical antenna consists of an extremely thin walled tube of infinite conductivity with an infinitesimally narrow circumferential gap. Physically, one expects the antenna current at the feedpoint of such a model to be infinite. However, the singularity in feedpoint current is not detected by either iterative or Fourier series solutions of the integral equation for antenna current. These solutions are continuous on zh, where h is the half-length of the antenna. They are also in good agreement with each other and with experimental data. From a formal point of view this amounts to saying that the conventional solutions of the integral equation are solutions on most, but not all, of the range zh. Inside a small region, zz_s, the correct solution to the integral equation is peculiar to the idealized generator. Low order iterative and Fourier series solutions ignore these peculiarities and produce a smooth current distribution which can be used in predicting the behavior of practical structures. This paper is concerned with a detailed study of the theoretical current near the feedpoint of the idealized model and the development of a definition of z_s.

The electric field at the ground plane near a disk-loaded monopole, J. Hansen and R. Larsen, *J. Research NBS* **66D** (*Radio Prop.*) *No. 2* (*Mar.–Apr. 1962*) *70 cents.*
In calculating ground losses for antennas with a ground wire system it is necessary to know the vertical electric field strength and the tangential magnetic field strength at the surface of the ground.
In this paper the vertical electric field strength at the ground plane near the base of an electrically short vertical antenna with a top loading in the shape of a circular disk is calculated. Numerical computations are carried out to some extent.

Propagation of plane electromagnetic waves past a shoreline, J. Bazer and S. N. Karp, *J. Research NBS* **66D** (*Radio Prop.*) *No. 3* (*May–June 1962*) *70 cents.*
The problems of the diffraction of homogeneous plane waves and ground waves by a linear shoreline in a planar land-sea surface are discussed. The direction of propagation of these incident waves is assumed perpendicular, and that of their magnetic vectors parallel, to the shoreline. At the air-land interface, the customary impedance boundary condition is imposed while the sea is treated as a perfect conductor; atmospheric and ionospheric effects are ignored. Exact integral representations of the solutions are presented. In the case of homogeneous plane-wave excitation originating over the sea, the integral representations are employed to obtain expressions for the geometrical optics field and for the far-field form of the remaining scattered field, transition regions included. The possibility of coastal refraction is discussed.

Currents induced on the surface of a conducting circular cylinder by a slot, G. Hasserjian and A. Ishimaru, *J. Research NBS* **66D** (*Radio Prop.*) *No. 3* (*May–June 1962*) *70 cents.*
This paper is a partial study of currents induced on circular, conducting cylinders by narrow radiating slots. First, a brief and general formulation of the radiation fields of slots on cylinders is made. Then, the problem of an infinite axial slot is examined thoroughly for all cylinder sizes. An expansion for the fields, very close to the slot, on large radius cylinders, is also obtained. Sample computations are made, for various ranges of cylinder radius, and the order of the errors is discussed.
The problem of a circumferential slot, of constant excitation, is also considered. An asymptotic expansion obtained for this case yields the surface current distribution for values of axial distances that are smaller than the square of the circumference of the cylinder.
Since one of the objectives of this study is to determine mutual coupling between two slots on a cylinder, the last section presents a formulation of the equivalent network in terms of the surface and feed line currents.

Periodicity modulo m and diversibility properties of the partition function, M. Newman, *Trans. Am. Math. Soc.* **97,** *No. 2, 225–236* (*Nov. 1960*).
It is proved that modulo 2, 5, 13 the unrestricted partition function $p(n)$ fills all residue classes infinitely often, and similar theorems for related functions are proved.

Applications of statistics in Post Office automation, B. M. Levin and N. C. Severo, *Am. Statistician* **15,** *No. 4, 14–18* (*Oct. 1961*).
The National Bureau of Standards is involved in designing and developing equipments and systems for the improved sorting of mail. As part of this project numerous statistical studies—mostly of a sampling type—have been conducted.

These studies were designed to obtain research and development information which is not collected by the Post Office as part of its routine data collecting activities. The data from these studies have proved useful as a basis for making administrative decisions, for developing efficient mechanized schemes of sorting mail, and for costing these schemes. This paper presents a thumbnail sketch of the Post Office mechanization problem, a description of the statistical problems encountered with some results of corresponding studies, and some of the uses to which the data have been put.

What is the best value? W. J. Youden, *J. Wash. Acad. Sci.* **51,** *No. 6, 95–97 (Oct. 1961).*
Investigators measuring physical constants frequently try various combinations of apparatus components and vary the operating conditions. Ideally none of these changes should affect the outcome. In fact very small effects are usually associated with many such alterations in the measurement procedure. A simple average of the various measurements may give undue weight to one or another of the substitutions when no basis exists for favoring one over the other. This note indicates how a planned program of changes avoids unequal weighing and increases the efficiency of detecting small effects.

Lower bounds for eigenvalues of Schrödinger's equation, N. W. Bazley and D. W. Fox, *Phys. Rev.* **124,** *No. 2, 483–492 (Oct. 1961).*
This paper gives new results that are useful in estimation of eigenvalues of Schrödinger's equation. Numerical applications are made for the helium atom, an anharmonic oscillator, and a radial Schrödinger equation.

Non-additivity in two-way analysis of variance, J. Mandel, *J. Am. Stat. Assoc.* **56,** *878–888 (Dec. 1961).*
In two-way classification analysis of variance situations there often exists a systematic type of row column interaction. A model is proposed in which the interaction is of the type $Q_i\gamma_j$ where Q_i is a parameter of the ith row, not necessarily associated with the main effect for rows, and γ_j is the main effect for column j. The analysis of data according to this model is given, including estimation and tests of significance. The model is more general than that involved in Tukey's "one degree of freedom for non-additivity." The relationship between the two methods is discussed. The application of the method to different types of problems is also discussed.

Advances in orthonormalizing computation, P. J. David and P. Rabinowitz, *Advances in Computers* **II,** *55–133 (1961).*
This paper contains: (1) a survey of least square approximation techniques in numerical analysis, (2) some recent numerical results on the solution of boundary value problems via the method of orthogonalized particular solutions.

Memory effects in irreversible thermodynamics, R. Zwanzig, *Phys. Rev.* **124,** *No. 4, 983–992 (Nov. 15, 1961).*
A new generalization of Onsager's theory of irreversible processes is presented. The main purpose is to allow for memory effects or causal time behavior, so that the response to a thermodynamic force comes later than the application of the force. This is accomplished by a statistical mechanical derivation of an exact non-Markoffian kinetic equation for the probability distribution in the space of macroscopic state variables. The memory effect in the resulting transport equations is represented by a time convolution of the thermodynamic forces with memory functions. The latter are time-correlation functions in the rates of change of the phase functions corresponding to macroscopic quantities. The resulting transport equations are not restricted to small deviations from thermal equilibrium. Onsager's theory is shown to be the low-frequency limit of our causal theory.

Formulae for an accurate intermediary orbit of an artificial satellite, J. P. Vinti, *Astronomical J.* **66,** *No. 9, 514–516 (Nov. 1961).*
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